SOLVABLE GROUPS

Consider an abelian group G.
Then $K(G) = \{e\}$
If $K(G)$ is abelian then

$$\{e\} = K(K(G)) \lhd K(G) \lhd G$$

We have a chain of commutants in general

$$\dots \lhd K_n(G) \lhd \dots K(K(G)) \lhd K(G) \lhd G$$

This chain may end with $\{e\}$, precisely when some commutant is abelian.
Does this always happen?
Example: Let $H, K \leq G$ s.t. $G = HK = \{hk \mid h \in H, k \in K\}$
Say H & K solvable & $H \lhd G$
Prove G solvable.

Claim: If $H \trianglelefteq G$ & (H & G/H) both solvable then G solvable

Proof: $\boxed{HW}$ Hint:

Consider $\varphi: G \to G/H$, the natural (surjective) hom.

So we will show G/H solvable.

Recall the third (second?) iso. thm.

"$H \triangleleft G$, $K < G$ then $HK < G$, $H \triangleleft HK$, $HK/H \cong K/(H \cap K)$"

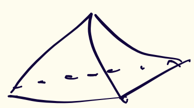
i.e. $G = HK$, $G/H \cong K/(H \cap K)$

Claim: G solvable & $N \triangleleft G$ then G/N solvable

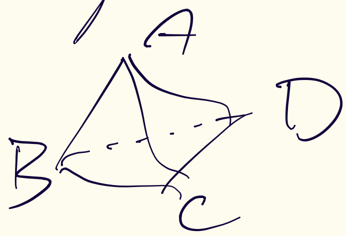
Proof: $\boxed{HW}$

K solvable so $K/(H \cap K)$ solvable so G/H solvable so G solvable.

$\boxed{Eg}$ Consider the gp. of rotations of a reg. tetrahedron



This is the group A_4. About every vertex there are 3 unique rotations (ACW). Write a permutations:



$$(BCD)(BCD) = (BDC)$$

$$A_4 = \{(BCD), (BDC), e, (ABC), (ACB), (ADC), (ACD), (ABD), (ADB), (AB)(CD), (AC)(BD), (AD)(BC)\}$$

& 3 rotations about opposite pairs of edges.

Commutant? $ab = ba$ for transp. $(ij)(kl) = (kl)(ij)$ iff $\{i,j\} \cap \{k,l\} = \emptyset$

$$(ijk)(ikj) = e \qquad (ijk)(ijl) = (il)(jk)$$

So $e \in K(A_4)$ & $(ij)(kl) \in K(A_4)$

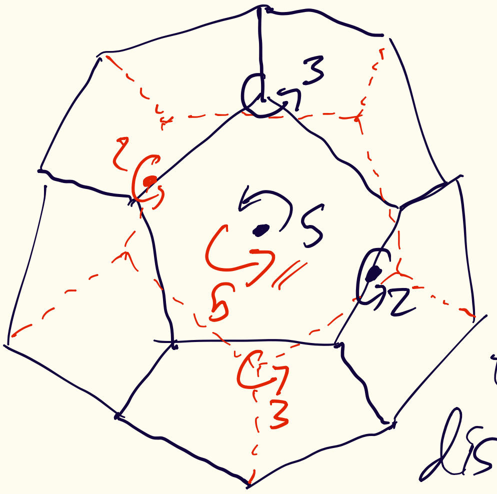
$$(ij)(kl)(ijk)(ij)(kl)(ikj) = (il)(kj)$$

So $K(A_4) = \{e, (AB)(CD), (AC)(BD), (AD)(BC)\}$

= Klein 4-group.

$\boxed{\text{HW}}$ Find the chain of commutants of S_4

$\boxed{\text{Eg}}$ The simplest geometric example is the regular dodecahedron, made up of 12 pentagons 6 pairs of opposing faces



10 pairs of opposing vertices
15 pairs of opposing edges
In each case, one rot. is the identity, which we discount, for now. So:

$6 \cdot (5-1) = 24$ rotations by faces $\to F$
$10 \cdot (3-1) = 20$ rotations by vertices $\to V$
$15 \cdot (2-1) = 15$ rotations by edges $\to E$
1 identity : Total is 60 rotations.

Note $|F \cup \{e\}| = 25 \nmid 60$
$|V \cup \{e\}| = 21 \nmid 60$
$|E \cup \{e\}| = 16 \nmid 60$

Let G be the group of rotations & let $N \triangleleft G$.
Let $F, E, \& V$ denote the rotations of faces, edges, and vertices respectively

Assume N contains some member of $F, E,$ or V
Say $a_1 \in F \cap N$. a_1 is a rotation about some face, $a_1 \neq e$ so all rotations about that face are in N (since 5 rotations & 5 is prime).
Let a be a rotation of $\boxed{\text{any}}$ other face.
There is some rotation in G sending the face of $a(f)$ to the face of $a_1(f_1)$, call it g.
Then gag^{-1} is a rotation of f_1!
$(gag^{-1}: f_1 \mapsto ga: f \mapsto g: f \mapsto f_1)$
i.e. $gag^{-1} = a_1^i$, some $i = 0, .., 4$
$gag^{-1} \in N \Rightarrow ga \in Ng \Rightarrow a \in g^{-1}Ng = N$
as $N \triangleleft G$

Hence N contains rotations of every face, & must contain <u>all</u> rotations of <u>all</u> faces.
A similar argument shows that if
$N \cap E \neq \emptyset$ then $E \subseteq N$
$N \cap V \neq \emptyset$ then $V \subseteq N$
Now: if $N \neq \{e\}$ then N contains at least one of $F, E,$ or V.

But: $|F| = 24$ $|E| = 15$ $|V| = 20$

& $c \in N$: No matter the combination, the number of elts of N will never divide 60 unless $|N| = 60$ i.e. G is <u>simple</u>.

<u>Lemma</u>: Let G be a simple non-abelian group. Then G is not soluble.

<u>Proof</u>: Recall $K(G) \trianglelefteq G$. Since G is not abelian, $K(G) \neq \{e\}$ and so $K(G) = G$.

Hence $K_n(G) = G \quad \forall n \in \mathbb{N}$. $\square$

In particular, the group of rotations of the dodecahedron is not soluble.

[In fact, the group of rotations of the dodecahedron is the same as the group of rotations of the 5-cell: A_5.]

(without proof)

Remark: A_5 is the smallest simple group that is not abelian. Recall abelian gps can be simple if they have no non-trivial subgps ($\mathbb{Z}_p$, p prime).